CS-466/566: Math for AI

Module 02: Computational Linear Algebra-2

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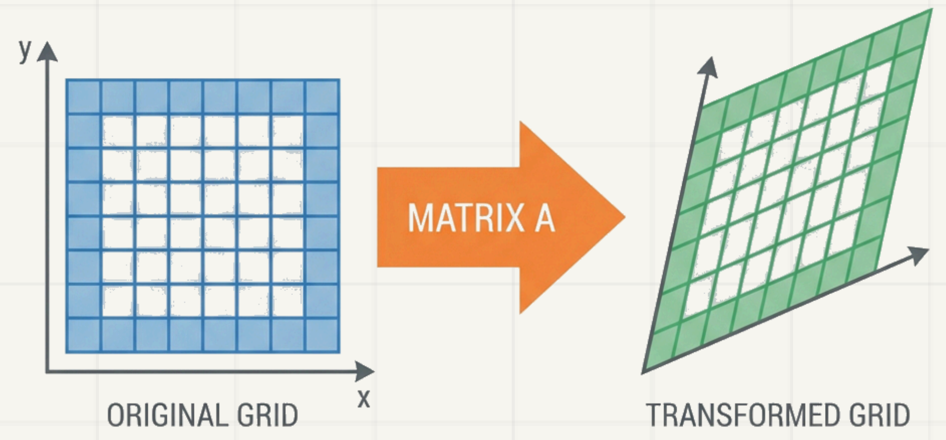
What is a Matrix?

1. The “Data” Perspective (CS):

A matrix is just a 2D array (table) of numbers.

$$X = \begin{bmatrix} \text{age} & \text{height} \\ 25 & 180 \\ 30 & 165 \\ 42 & 175 \end{bmatrix}$$

- **Rows**: Samples (people)
- **Columns**: Features (attributes)



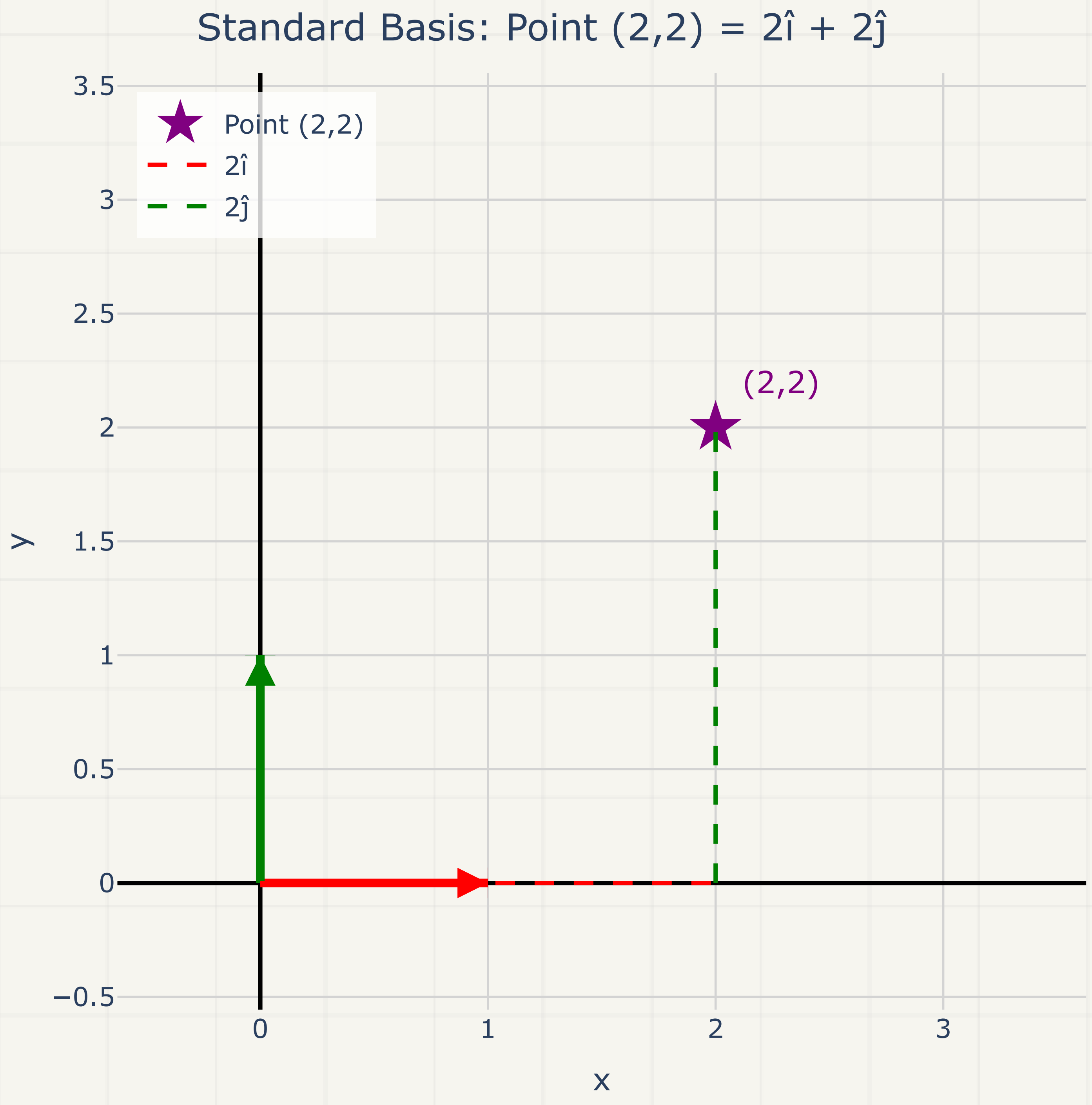
We will focus on the **Math Perspective** today.

2. The “Math” Perspective:

A matrix is a **function** or **transformation**.

- It acts on vectors: $f(\mathbf{x}) = A\mathbf{x}$
- It warps space (stretches, rotates, shears).

Matrices as Basis Changers (1/2)



1. The Standard World (Identity)

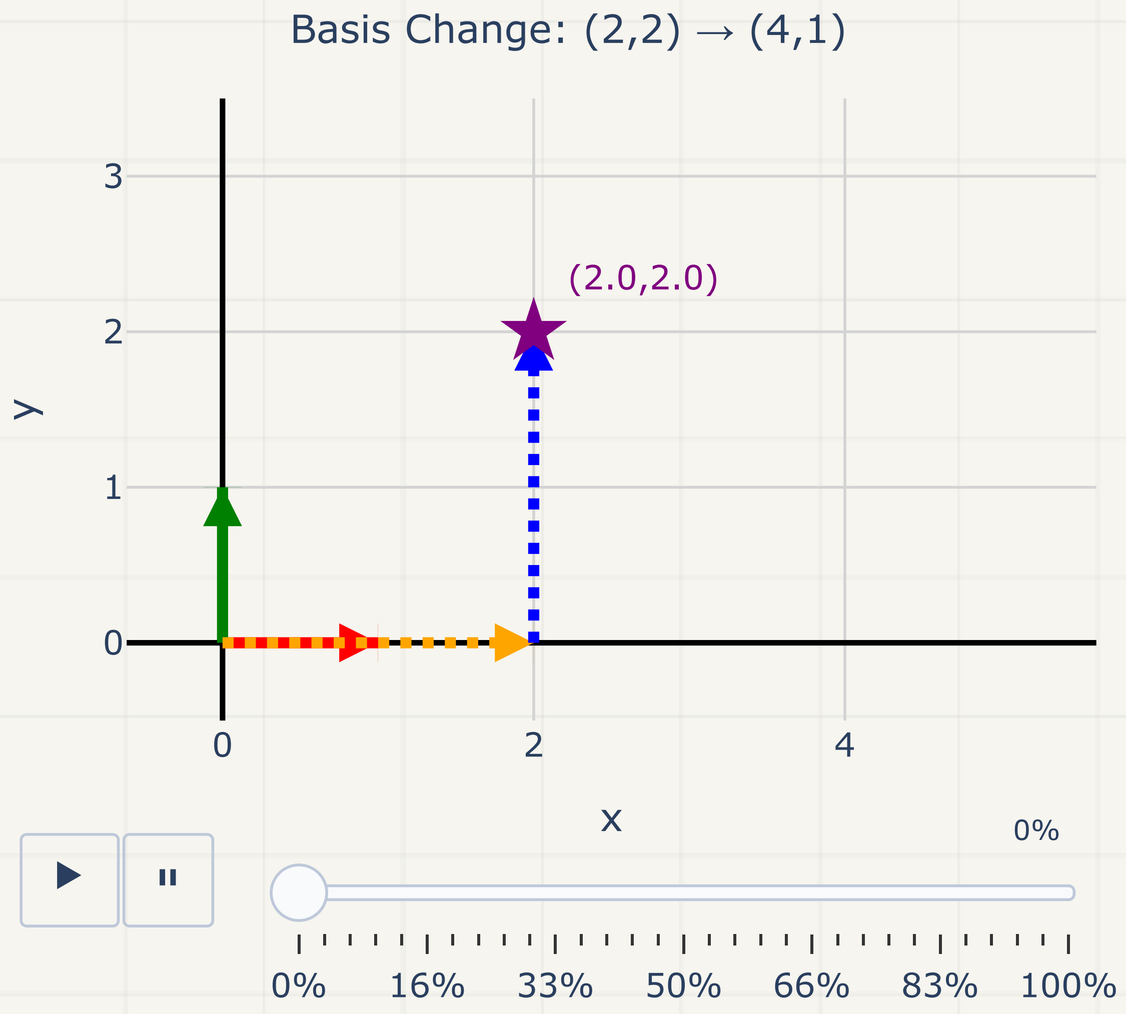
$$I = [\mathbf{e}_1 | \mathbf{e}_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A vector $\mathbf{x} = (2, 2)$ means:

$$\mathbf{x} = 2\mathbf{e}_1 + 2\mathbf{e}_2 = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Result: You land exactly where you expect, at $(2, 2)$.

Matrices as Basis Changers (2/2)



2. The Transformed World

$$A = [\mathbf{v}_1 | \mathbf{v}_2] = \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix}$$

The **same** coordinates $(2, 2)$ now mean:

$$A\mathbf{x} = 2\mathbf{v}_1 + 2\mathbf{v}_2 = 2\begin{bmatrix} 2 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Result: The numbers (2,2) now land you at the physical location $(4, 1)$.

Matrix-Vector Multiplication is just a **Linear Combination of Columns**.

Matrix-Vector Multiplication: The Mechanics

The mechanics of matrix-vector multiplication can be visualized in two ways:

Column View

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$= 2 \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + 0 \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 4 \\ 13 \\ 22 \end{bmatrix}$$

Row View (Dot Products)

Row 1: $[1, 2, 3] \cdot [2, 1, 0] = 4$

Row 2: $[4, 5, 6] \cdot [2, 1, 0] = 13$

Row 3: $[7, 8, 9] \cdot [2, 1, 0] = 22$

$$\Rightarrow \begin{bmatrix} 4 \\ 13 \\ 22 \end{bmatrix}$$

Both views give the same result — use whichever is more intuitive!

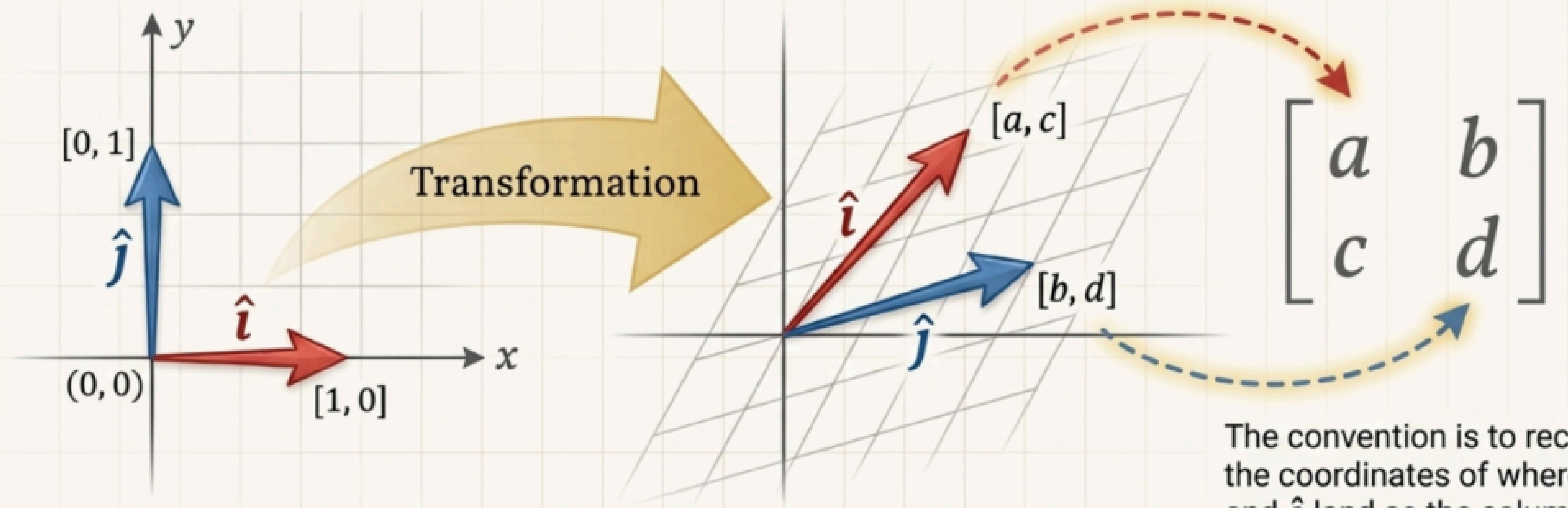
Types of Matrix Transformations

Type	Matrix	Effect	Example: $A \times (2,2)^T$
Identity	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	No change	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$
Scaling	$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$	Stretch	$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$
Shear	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	Slant	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$
Reflection	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	Flip	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$
Rotation 90°	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	Rotate	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$

Combining transformations: $C = BA$ applies A first, then B

Matrix Columns = Transformed Basis Vectors

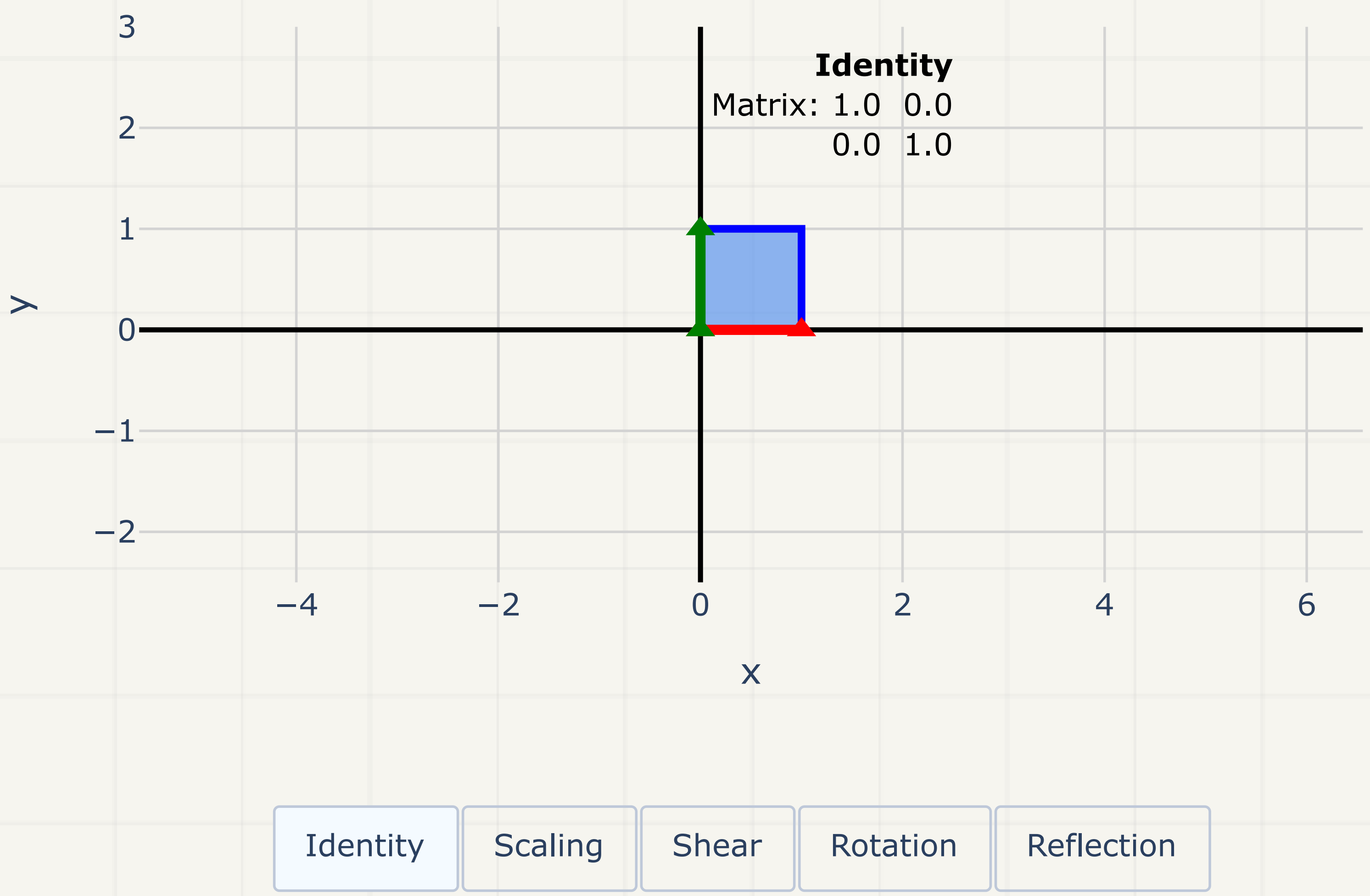
A linear transformation is completely determined by where it takes the basis vectors. Any other vector's destination can be found from there.



The convention is to record the coordinates of where $\hat{\imath}$ and $\hat{\jmath}$ land as the columns of a matrix.

The columns of a matrix tell you where $\hat{i}$ and $\hat{j}$ land!

Matrix Transformations in Action



Matrix columns show where basis vectors $\hat{i}$ and $\hat{j}$ land after the transformation!

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What is Matrix Multiplication?

A acts on **each column** of B to produce each column of C

$$A \cdot B = A \cdot \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{bmatrix} = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_n \end{bmatrix} = C$$

Example:

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \left(A \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad A \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

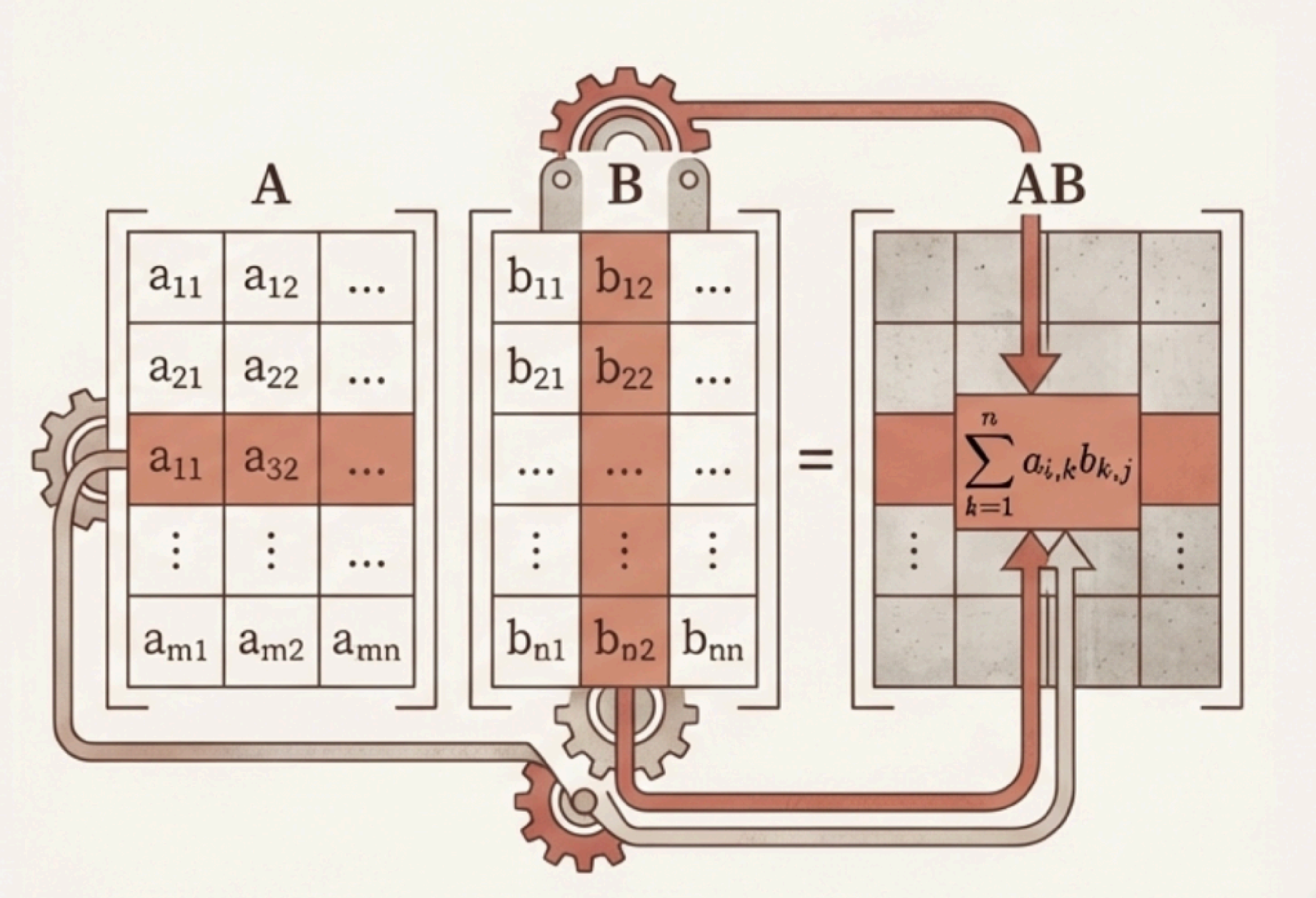
Each column of the identity is transformed by the scaling matrix!

Matrix multiplication = applying the left matrix to each column of the right matrix.

Matrix Multiplication: The Mechanics

Computing $(AB)_{ij}$ **:**

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$$



Steps:

1. Take **row** i from A
2. Take **column** j from B
3. **Dot product** → element (i, j) of C

Dimension rule:

$$(m \times n) \cdot (n \times p) = (m \times p)$$

Row of A meets Column of B → one element of AB

Exercise: Matrix Multiplication

Calculate $C = A \times B$:

$$\underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{2\times3} \times \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}}_{3\times2}$$

1. What are the dimensions of the output matrix C ?

Answer: 2×2

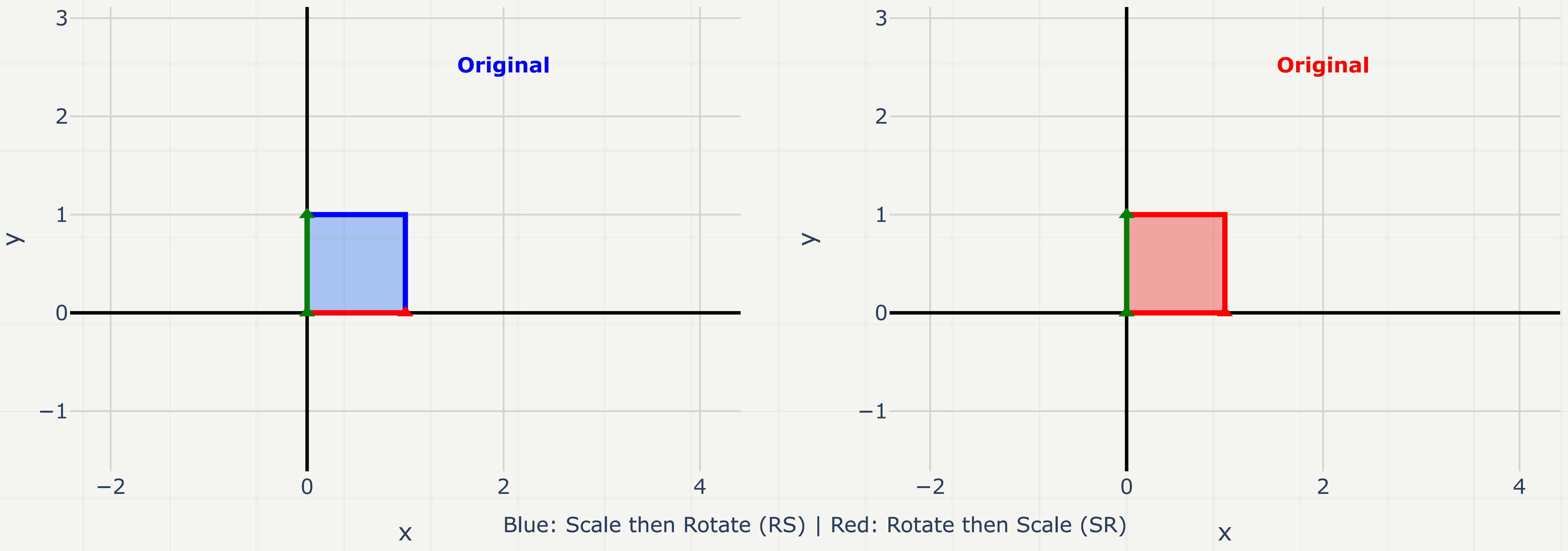
(Inner dimensions 3 match; Outer dimensions 2×2 remain)

2. What is the result matrix C ?

Answer:

$$\begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}$$

Matrix Multiplication: Function Composition - Order Matters



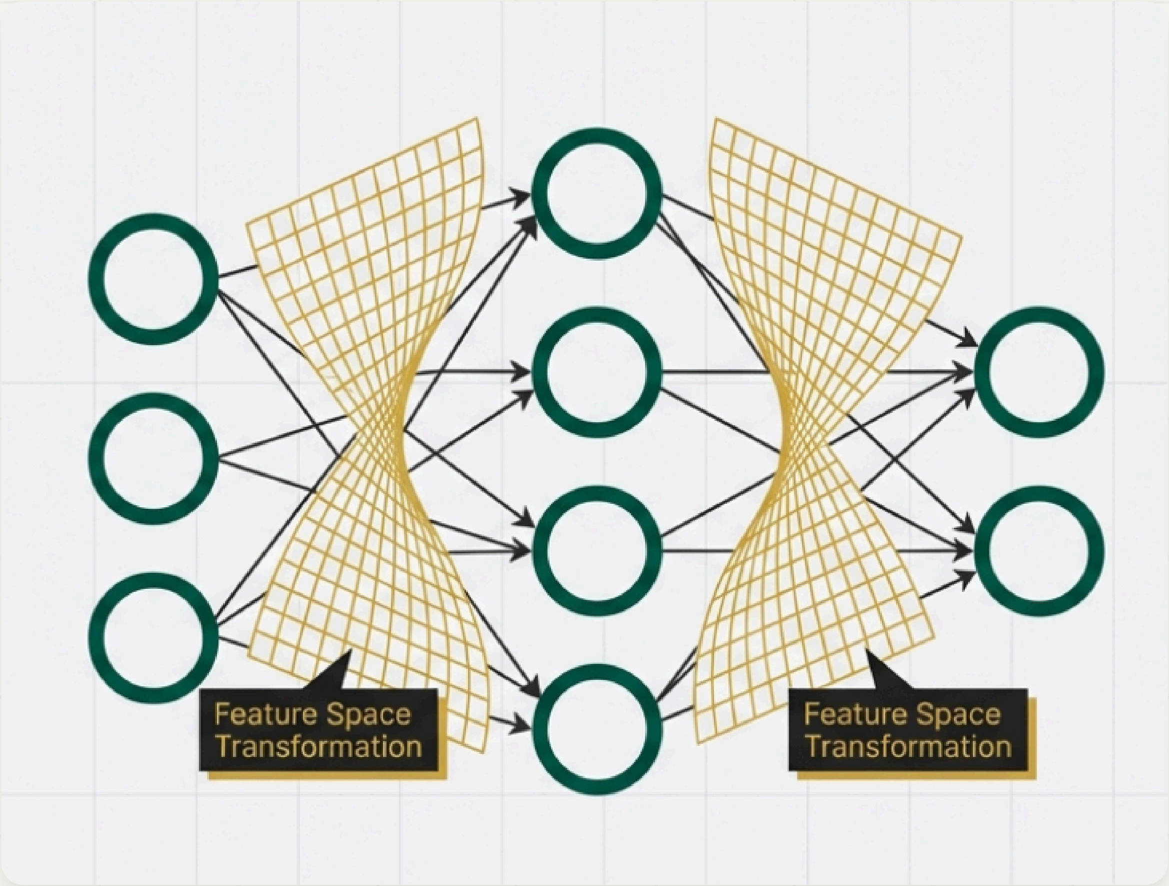
Blue: Scale then Rotate (RS) | Red: Rotate then Scale (SR)

Original | First Transform | Composition

Scale then Rotate (left): $RS = \underbrace{\begin{pmatrix} 0.7 & -0.7 \\ 0.7 & 0.7 \end{pmatrix}}_{R} \underbrace{\begin{pmatrix} 2 & 0 \\ 0 & 0.5 \end{pmatrix}}_{S} = \begin{pmatrix} 1.4 & -0.35 \\ 1.4 & 0.35 \end{pmatrix}$

Rotate then Scale (right): $SR = \begin{pmatrix} 2 & 0 \\ 0 & 0.5 \end{pmatrix} \begin{pmatrix} 0.7 & -0.7 \\ 0.7 & 0.7 \end{pmatrix} = \begin{pmatrix} 1.4 & -1.4 \\ 0.35 & 0.35 \end{pmatrix}$

Machine Learning Connection: Neural Networks



In Deep Learning, each layer of a Neural Network is just a **matrix multiplication** (plus a non-linearity).

- The matrix **warps the space** (shears, rotates, stretches).
- The goal? Transform “tangled” data into a space where it is **linearly separable** (easy to classify).

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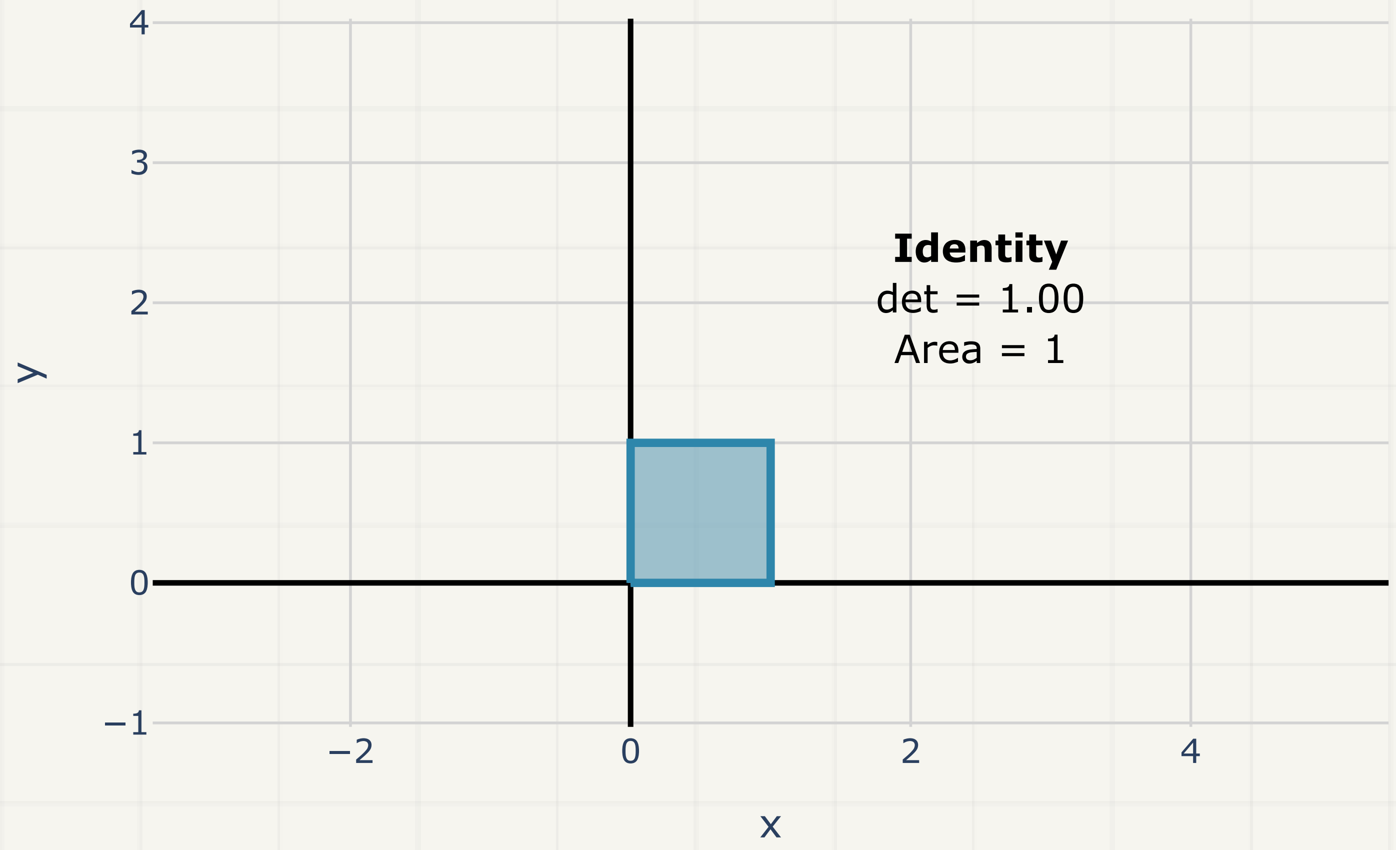
What is the Determinant?



Identity | Scale 2×3 | Scale 0.5×2 | Shear | Reflection

Watch how different transformations change the area of the unit square!

What is the Determinant?

The determinant measures how much a transformation scales area.

For a 2×2 matrix:

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

Geometric Meaning:

- Start with a unit square (area = 1)
- Apply matrix transformation
- New area = $|\det(A)|$

Quick Examples:

Matrix	det	Area Factor
$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$	6	6× bigger
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	1	Same area
$\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$	0	Collapsed!

The determinant tells you how the transformation scales space!

What the Sign Tells Us

det > 0

✅ Orientation Preserved

The “handedness” stays the same (counterclockwise stays counterclockwise)

det < 0

🔄 Orientation Flipped

Like looking in a mirror — left becomes right

det = 0

💀 Space Collapsed

2D → 1D line (or point). Matrix is **singular** — no inverse!

det = 0 means the transformation loses information — you can’t undo it!

Computing Determinants in NumPy

```python
import numpy as np

#> Define matrices
A = np.array([[2, 0],
              [0, 3]])    # Scaling matrix

B = np.array([[1, 2],
              [3, 4]])    # General matrix

C = np.array([[2, 4],
              [1, 2]])    # Singular matrix (det = 0)

#> Compute determinants
print(f"det(A) = {np.linalg.det(A):.2f}")  #> Expected: 6
print(f"det(B) = {np.linalg.det(B):.2f}")  #> Expected: -2
print(f"det(C) = {np.linalg.det(C):.2f}")  #> Expected: 0
```

```
det(A) = 6.00
det(B) = -2.00
det(C) = 0.00
```

Use `np.linalg.det(A)` to compute the determinant of any matrix!

Exercise: Determinants

Calculate the determinant of $M = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$.

1. Compute $\det(M) = ad - bc$:

Answer: 10

($3(4) - 2(1) = 12 - 2 = 10$)

2. Geometric Interpretation:

Answer: Area scales by 10x

The unit square becomes a parallelogram with area 10.

Exercise: Determinant

Are the vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ linearly independent? Use the determinant!

1. Form Matrix & Compute Det:

$$\det \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} = 1(4) - 2(2) = 0$$

2. Conclusion:

Linearly Dependent!

Since $\det = 0$, the “area” is zero. The vectors lie on the same line (they are parallel).

What is a Matrix Inverse?

The inverse A^{-1} **"undo"** the transformation A :

$$AA^{-1} = A^{-1}A = I$$

Geometric Intuition

- **Rotation:** Rotate 30° $\longrightarrow$ Rotate -30°
- **Scaling:** Scale by 2 $\longrightarrow$ Scale by 0.5
- **Shear:** Shear right $\longrightarrow$ Shear left

2×2 Inverse Formula

$$A^{-1} = \frac{1}{\det(A)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Swap a, d | **Negate** b, c | **Divide** by $\det$

The inverse only exists when $\det(A) \neq 0$!

Matrix Inverse: Example

Given:

$$A = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$$

Step 1: Calculate determinant

$$\det(A) = 3 \times 4 - 1 \times 2 = 12 - 2 = 10$$

Step 2: Apply the formula (swap, negate, divide)

$$A^{-1} = \frac{1}{10}\begin{pmatrix} 4 & -1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 0.4 & -0.1 \\ -0.2 & 0.3 \end{pmatrix}$$

Step 3: Verify: $A \cdot A^{-1} = I$

Always verify your inverse by checking that $A \cdot A^{-1} = I$!

Computing Inverses in NumPy

```python
import numpy as np
#> Define a matrix
A = np.array([[3, 1],
              [2, 4]])
#> Compute inverse
A_inv = np.linalg.inv(A)
print("\nA^(-1) =\n", A_inv)

#> Verify: A @ A_inv = I
print("\nA @ A^(-1) =\n", A @ A_inv)
```

```

A^(-1) =
 [[ 0.4 -0.1]
 [-0.2  0.3]]

A @ A^(-1) =
 [[1. 0.]
 [0. 1.]]
```

Use `np.linalg.inv(A)` — but only if you really need the inverse!

What is Matrix Rank?

What is Matrix Rank?

The **rank** of a matrix is the number of **linearly independent** rows (or columns).

Intuitive View

Rank = "True Dimensionality"

- How many independent directions does the matrix span?
- A 3×3 matrix with rank 2 only spans a 2D plane
- Rank tells you the "effective size" of the transformation

Key Facts

- $\text{rank}(A) \leq \min(m, n)$ for $m \times n$ matrix
- Row rank = Column rank (always!)
- **Full rank:** $\text{rank}(A) = \min(m, n)$

Rank measures how much "information" a matrix truly contains!

Rank: The Geometric View

Rank = 2 (Full)

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Maps 2D → 2D

✅ Spans the whole plane

Rank = 1 (Deficient)

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

Maps 2D → 1D line

⚠️ Columns are parallel

Rank = 0

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Maps everything → origin

💀 No information preserved

Lower rank = Lost dimensions = Lost information!

Rank and Key Concepts

Concept	Connection to Rank
Determinant	$\det(A) \neq 0 \Leftrightarrow$ full rank (for square matrices)
Inverse	A^{-1} exists $\Leftrightarrow$ full rank
Linear Independence	Columns are independent $\Leftrightarrow$ full column rank
Null Space	$\dim(\text{null}(A)) = n - \text{rank}(A)$
Solutions to $Ax = b$	Unique solution $\Leftrightarrow$ full rank

The Rank-Nullity Theorem

$$\text{rank}(A) + \text{nullity}(A) = n$$

(# of independent columns) + (# of free variables) = (total # of columns)

Computing Rank in NumPy

```python
import numpy as np

#> Full rank matrix (rank = 2)
A = np.array([[1, 2],
              [3, 4]])

#> Rank-deficient matrix (rank = 1)
B = np.array([[1, 2],
              [2, 4]])  # Row 2 = 2 × Row 1

#> Compute ranks
print(f"rank(A) = {np.linalg.matrix_rank(A)}")  #> Expected: 2
print(f"rank(B) = {np.linalg.matrix_rank(B)}")  #> Expected: 1

#> Verify with determinant
print(f"\ndet(A) = {np.linalg.det(A):.2f}")  #> Non-zero (invertible)
print(f"det(B) = {np.linalg.det(B):.2f}")    #> Zero (singular)
```

```
rank(A) = 2
rank(B) = 1

det(A) = -2.00
```

Exercise: Matrix Rank

Find the rank of $M = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 1 \end{pmatrix}$.

1. Check for Linear Dependence:

Row 2 = 2 × Row 1

Rows 1 and 2 are linearly dependent — we effectively have only 2 independent rows.

2. What is the Rank?

Answer: rank(M) = 2

Only 2 linearly independent rows (Row 1 and Row 3).

3. Is M Invertible?

No!

A 3×3 matrix needs rank 3 to be invertible. Since rank(M) = 2 < 3, the matrix is singular.

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Change of Basis

Why Change Basis?

Different coordinate systems describe **the same vector** differently.

Example: A point at $(3, 2)$ in standard coordinates might be $(1, 1)$ in a rotated coordinate system!

Same point in space, different numbers to describe it!

Intuition: Translating Instructions

Think of coordinates as **instructions** to reach a point.

- **Standard Basis** (I): "Go 4 East, 3 North".
- **New Basis** (B): "Go ? along vector $\mathbf{b}_1$, ? along vector $\mathbf{b}_2$".

To find the new instructions, we need to **undo** the shape change caused by B.

Multiply by B^{-1} to "unwrap" the transformation!

The Change of Basis Formula

If B contains new basis vectors as **columns**:

To convert TO new basis:

$$[\mathbf{v}]_{\text{new}} = B^{-1}[\mathbf{v}]_{\text{standard}}$$

To convert FROM new basis:

$$[\mathbf{v}]_{\text{standard}} = B[\mathbf{v}]_{\text{new}}$$

Key Insight: B^{-1} converts TO the new basis, B converts FROM it.

The inverse of a basis matrix converts coordinates between systems!

Change of Basis: Example

Problem: Convert $\mathbf{v} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ to a stretched basis.

New basis: $\mathbf{b}_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, $\mathbf{b}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Step 1: Build basis matrix $B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

Step 2: Find inverse $B^{-1} = \begin{pmatrix} 0.5 & 0 \\ 0 & 1 \end{pmatrix}$

Step 3: Convert: $[\mathbf{v}]_{\text{new}} = B^{-1}\mathbf{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

Step 4: Verify: $2\mathbf{b}_1 + 3\mathbf{b}_2 = 2\begin{pmatrix} 2 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \mathbf{v}$

✅

Exercise: Change of Basis

Convert $\mathbf{v} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ to the basis □ $= \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$.

1. Form the Basis Matrix B:

Answer: $B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

2. Find the Inverse B^{-1} (Hint: $\det(B) = 2$):

Answer: $B^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

3. Compute New Coordinates $[\mathbf{v}]_{\text{new}} = B^{-1}\mathbf{v}$:

Answer: $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

4. Verify your answer:

Check: $3\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 1\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ ✅

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What is a Linear System?

Find $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$

System of Equations

$$\begin{cases} 2x + y = 5 \\ x + 3y = 7 \end{cases}$$

Matrix Form

$$\underbrace{\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} 5 \\ 7 \end{pmatrix}}_{\mathbf{b}}$$

The Key Question

"What linear combination of columns of A gives us $\mathbf{b}$?"

$$x_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$$

Determinant and Solutions

One Solution	Infinite Solutions	No Solution
✅ $\det(A) \neq 0$	⚠️ $\det(A) = 0$, consistent	💀 $\det(A) = 0$, inconsistent
$\begin{cases} x + y = 3 \\ x - y = 1 \end{cases}$	$\begin{cases} x + y = 2 \\ 2x + 2y = 4 \end{cases}$	$\begin{cases} x + y = 2 \\ x + y = 5 \end{cases}$

Zero determinant means either no solution or infinite solutions.

Solving $A\mathbf{x} = \mathbf{b}$: Two Methods

❌ Using the Inverse

$$\mathbf{x} = A^{-1}\mathbf{b}$$

Problems?

- Computing A^{-1} is expensive: $O(n^3)$
- Numerically unstable
- Only works for square, invertible A

✅ Direct Methods

Gaussian Elimination / LU Decomposition

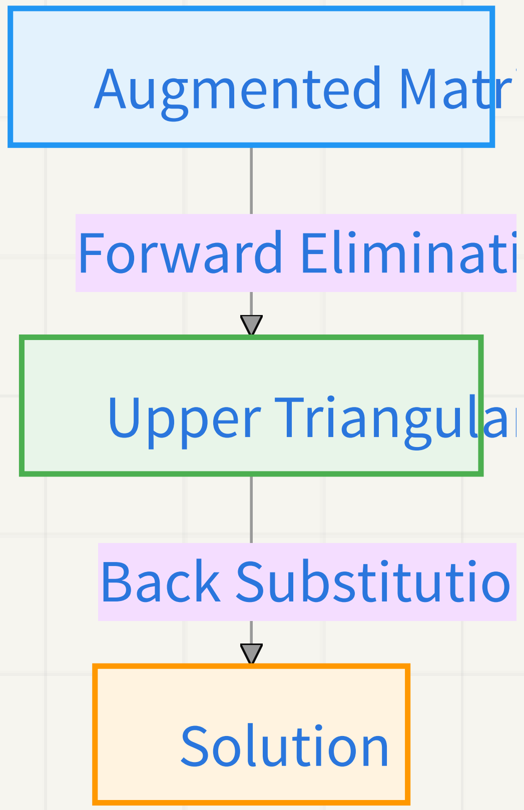
Advantages?

- More efficient: same $O(n^3)$ but smaller constant
- More stable numerically
- `np.linalg.solve()` uses this!

Never compute A^{-1} just to solve $A\mathbf{x} = \mathbf{b}$!

Gaussian Elimination: The Algorithm

Goal: Transform the system into an upper-triangular form ("Row Echelon Form") so we can solve it easily from bottom to top.



Visualizing Row Operations

Solve for x, y :

$$\begin{cases} 2x + y = 5 \\ 4x - y = 1 \end{cases} \implies \left[\begin{array}{cc|c} 2 & 1 & 5 \\ 4 & -1 & 1 \end{array}\right]$$

Step 1: Eliminate the 4 (make it 0). Target Row 2. Operation: $R_2 \leftarrow R_2 - 2R_1$.

$$\left[\begin{array}{cc|c} 2 & 1 & 5 \\ 4 & -1 & 1 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|c} 2 & 1 & 5 \\ 0 & -3 & -9 \end{array}\right]$$

Now it's Upper Triangular!

Back Substitution

From the triangular matrix:

$$\left[\begin{array}{cc|c} 2 & 1 & 5 \\ 0 & -3 & -9 \end{array}\right]$$

1. Solve bottom equation first:

$$-3y = -9 \implies y = 3$$

2. Substitute into top equation:

$$2x + y = 5 \implies 2x + 3 = 5 \implies 2x = 2 \implies x = 1$$

Solution: $\mathbf{x}, \mathbf{y} = (1, 3)$

Gaussian Elimination: 3x3 System

$$\begin{cases} x + y + z = 6 \\ 2x + 4y + 2z = 16 \\ -x + 5y - 4z = -3 \end{cases} \implies \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & 4 & 2 & 16 \\ -1 & 5 & -4 & -3 \end{array}\right]$$

Step 1: Clear Column 1 (below pivot) $R_2 \leftarrow R_2 - 2R_1, \quad R_3 \leftarrow R_3 + R_1$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 2 & 0 & 4 \\ 0 & 6 & -3 & 3 \end{array}\right]$$

Step 2: Clear Column 2 (below pivot) $R_3 \leftarrow R_3 - 3R_2$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & -3 & -9 \end{array}\right]$$

Back Substitution: $z = 3 \implies y = 2 \implies x = 1$

Computing Inverse via Gaussian Elimination

Core Idea: Augment A with I, row reduce until $A \to I$. Then right side is A^{-1}.

$$[A|I] \xrightarrow{\text{RREF}} [I|A^{-1}]$$

1. Setup $[A|I]$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right]$$

2. Eliminate

$R_2 \leftarrow R_2 - 3R_1$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right]$$

3. Solve & Normalize

R_1, R_2 ops

$$\left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 1.5 & -0.5 \end{array}\right]$$

Result: $A^{-1} = \begin{pmatrix} -2 & 1 \\ 1.5 & -0.5 \end{pmatrix}$

Exercise: 3x3 Inverse

Compute the inverse of $A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}$

1. Setup Augmented Matrix:

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{array}\right]$$

2. Row Reduce ($A \longrightarrow I$): - Elim Col 1: $R_2 - 2R_1$, $R_3 - 4R_1$ - Elim Col 2: $R_3 + R_2$ - Normalize & Back-Sub

Answer: $A^{-1} = \begin{pmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{pmatrix}$

Exercise: Solving Linear Systems

Solve the system: $\begin{cases} 3x + 2y = 12 \\ x + 4y = 10 \end{cases}$

1. Write in Matrix Form $A\mathbf{x} = \mathbf{b}$:

Answer: $\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 12 \\ 10 \end{pmatrix}$

2. Check: Is there a unique solution? (Hint: compute $\det(A)$)

Answer: $\det(A) = 3(4) - 2(1) = 10 \neq 0$ → Yes, unique solution!

3. Solve using $\mathbf{x} = A^{-1}\mathbf{b}$:

Answer: $A^{-1} = \frac{1}{10}\begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix}$ → $\mathbf{x} = \begin{pmatrix} 2.8 \\ 1.8 \end{pmatrix}$

Solving Linear Systems in NumPy

```python
import numpy as np

#> Define the system: Ax = b
A = np.array([[2, 1],
              [1, 3]])
b = np.array([5, 7])

#> Method 1: Using solve (RECOMMENDED)
x = np.linalg.solve(A, b)
print(f"Solution: x = {x}")
#> Verify: A @ x should equal b
print(f"Verification: A @ x = {A @ x}")
#> Method 2: Using inverse (NOT recommended)
x_inv = np.linalg.inv(A) @ b
print(f"Via inverse: x = {x_inv}")
```

```
Solution: x = [1.6 1.8]
Verification: A @ x = [5. 7.]
Via inverse: x = [1.6 1.8]
```

Always use `np.linalg.solve(A, b)` — it's faster and more accurate!

Thank You!